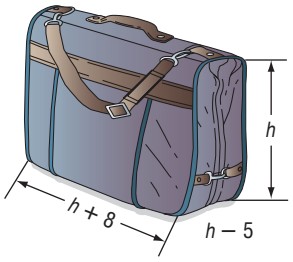
6-9 Rational Zero Theorem

Main Ideas

- Identify the possible rational zeros of a polynomial function.
- Find all the rational zeros of a polynomial function.

GET READY for the Lesson

On an airplane, carry-on baggage must fit into the overhead compartment above the passenger's seat. The length of the compartment is 8 inches longer than the height, and the width is 5 inches shorter than the height. The volume of the compartment is 2772 cubic inches. You can solve the polynomial equation $h(h + 8)(h - 5) = 2772$, where h is the height, $h + 8$ is the length, and $h - 5$ is the width, to find the dimensions of the overhead compartment.



Identify Rational Zeros Usually it is not practical to test all possible zeros of a polynomial function using only synthetic substitution. The **Rational Zero Theorem** can help you choose some possible zeros to test.

Study Tip

The Rational Zero Theorem only applies to rational zeros. Not *all* of the roots of a polynomial are found using the divisibility of the coefficients.

KEY CONCEPT *Rational Zero Theorem*

Words Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then p is a factor of a_0 and q is a factor of a_n.

Example Let $f(x) = 2x^3 + 3x^2 - 17x + 12$. If $\frac{3}{2}$ is a zero of $f(x)$, then 3 is a factor of 12 and 2 is a factor of 2.

In addition, if the coefficient of the x term with the highest degree is 1, we have the following corollary.

KEY CONCEPT *Corollary (Integral Zero Theorem)*

If the coefficients of a polynomial function are integers such that $a_n = 1$ and $a_0 \neq 0$, any rational zeros of the function must be factors of a_n.

EXAMPLE 1 **Identify Possible Zeros**

List all of the possible rational zeros of each function.

a. $f(x) = 2x^3 - 11x^2 + 12x + 9$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 9 and q is a factor of 2. The possible values of p are ±1, ±3, and ±9. The possible values for q are ±1 and ±2. So, $\frac{p}{q} = \pm1, \pm3, \pm9, \pm\frac{1}{2}, \pm\frac{3}{2}$, and $\pm\frac{9}{2}$.

(continued on the next page)

b. $f(x) = x^3 - 9x^2 - x + 105$

Since the coefficient of x^3 is 1, the possible rational zeros must be a factor of the constant term 105. So, the possible rational zeros are the integers ± 1, ± 3, ± 5, ± 7, ± 15, ± 21, ± 35, and ± 105.

CHECK Your Progress

1A. $g(x) = 3x^3 - 4x + 10$ **1B.** $h(x) = x^3 + 11x^2 + 24$

Find Rational Zeros Once you have found the possible rational zeros of a function, you can test each number using synthetic substitution to determine the zeros of the function.

EXAMPLE **Find Rational Zeros**

2 **GEOMETRY** **The volume of a rectangular solid is 675 cubic centimeters. The width is 4 centimeters less than the height, and the length is 6 centimeters more than the height. Find the dimensions of the solid.**

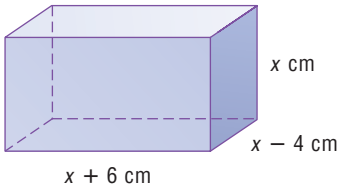
Let x = the height, $x - 4$ = the width, and $x + 6$ = the length.

Study Tip

Descartes' Rule of Signs

Examine the signs of the coefficients in the equation, $+ + - -$. There is one change of sign, so there is only one positive real zero.

Write an equation for the volume.



$$\ell wh = V \quad \text{Formula for volume}$$
$$(x - 4)(x + 6)x = 675 \quad \text{Substitute.}$$
$$x^3 + 2x^2 - 24x = 675 \quad \text{Multiply.}$$
$$x^3 + 2x^2 - 24x - 675 = 0 \quad \text{Subtract 675.}$$

p	1	2	−24	−675
1	1	3	−21	−696
3	1	5	−9	−702
5	1	7	11	−620
9	1	11	75	0

The leading coefficient is 1, so the possible integer zeros are factors of 675, ± 1, ± 3, ± 5, ± 9, ± 15, ± 25, ± 27, ± 45, ± 75, ± 135, ± 225, and ± 675. Since length can only be positive, we only need to check positive zeros. From Descartes' Rule of Signs, we also know there is only one positive real zero. Make a table for the synthetic division and test possible real zeros.

One zero is 9. Since there is only one positive real zero, we do not have to test the other numbers. The other dimensions are $9 - 4$ or 5 centimeters and $9 + 6$ or 15 centimeters.

CHECK Verify that the dimensions are correct. $5 \times 9 \times 15 = 675$ ✓

CHECK Your Progress

2. The volume of a rectangular solid is 1056 cubic inches. The length is 1 inch more than the width, and the height is 3 inches less than the width. Find the dimensions of the solid.

You usually do not need to test all of the possible zeros. Once you find a zero, you can try to factor the depressed polynomial to find any other zeros.

EXAMPLE 3 Find All Zeros

Find all of the zeros of $f(x) = 2x^4 - 13x^3 + 23x^2 - 52x + 60$.

From the corollary to the Fundamental Theorem of Algebra, we know there are exactly 4 complex roots. According to Descartes' Rule of Signs, there are 4, 2, or 0 positive real roots and 0 negative real roots. The possible rational zeros are ±1, ±2, ±3, ±4, ±5, ±6, ±10, ±12, ±15, ±20, ±30, ±60, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, and $\pm\frac{15}{2}$. Make a table and test some possible rational zeros.

$\frac{p}{q}$	2	−13	23	−52	60
1	2	−11	12	−40	20
2	2	−9	5	−42	−24
3	2	−7	2	−46	−78
5	2	−3	8	−12	0

Since $f(5) = 0$, you know that $x = 5$ is a zero. The depressed polynomial is $2x^3 - 3x^2 + 8x - 12$.

Factor $2x^3 - 3x^2 + 8x - 12$.

$2x^3 - 3x^2 + 8x - 12 = 0$ Write the depressed polynomial.

$2x^3 + 8x - 3x^2 - 12 = 0$ Regroup terms.

$2x(x^2 + 4) - 3(x^2 + 4) = 0$ Factor by grouping.

$(x^2 + 4)(2x - 3) = 0$ Distributive Property

$x^2 + 4 = 0$ or $2x - 3 = 0$ Zero Product Property

$x^2 = -4$ $\qquad 2x = 3$

$x = \pm 2i$ $\qquad x = \frac{3}{2}$

There is another real zero at $x = \frac{3}{2}$ and two imaginary zeros at $x = 2i$ and $x = -2i$.

The zeros of this function are 5, $\frac{3}{2}$, $2i$ and $-2i$.

CHECK Your Progress

Find all of the zeros of each function.

3A. $h(x) = 9x^4 + 5x^2 - 4$ **3B.** $k(x) = 2x^4 - 5x^3 + 20x^2 - 45x + 18$

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CHECK Your Understanding

Example 1 (pp. 369–370)

List all of the possible rational zeros of each function.

1. $p(x) = x^4 - 10$ **2.** $d(x) = 6x^3 + 6x^2 - 15x - 2$

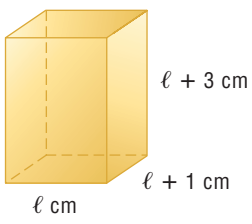
Example 2 (p. 370)

Find all of the rational zeros of each function.

3. $p(x) = x^3 - 5x^2 - 22x + 56$ **4.** $f(x) = x^3 - x^2 - 34x - 56$

5. $t(x) = x^4 - 13x^2 + 36$ **6.** $f(x) = 2x^3 - 7x^2 - 8x + 28$

7. GEOMETRY The volume of the rectangular solid is 1430 cubic centimeters. Find the dimensions of the solid.



Example 3 (p. 371)

Find all of the zeros of each function.

8. $f(x) = 6x^3 + 5x^2 - 9x + 2$

9. $f(x) = x^4 - x^3 - x^2 - x - 2$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–15	1
16–21	2
22–29	3

List all of the possible rational zeros of each function.

10. $f(x) = x^3 + 6x + 2$

11. $h(x) = x^3 + 8x + 6$

12. $f(x) = 3x^4 + 15$

13. $n(x) = x^5 + 6x^3 - 12x + 18$

14. $p(x) = 3x^3 - 5x^2 - 11x + 3$

15. $h(x) = 9x^6 - 5x^3 + 27$

Find all of the rational zeros of each function.

16. $f(x) = x^3 + x^2 - 80x - 300$

17. $p(x) = x^3 - 3x - 2$

18. $f(x) = 2x^5 - x^4 - 2x + 1$

19. $f(x) = x^5 - 6x^3 + 8x$

20. $g(x) = x^4 - 3x^3 + x^2 - 3x$

21. $p(x) = x^4 + 10x^3 + 33x^2 + 38x + 8$

Find all of the zeros of each function.

22. $p(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40$

23. $g(x) = 5x^4 - 29x^3 + 55x^2 - 28x$

24. $h(x) = 6x^3 + 11x^2 - 3x - 2$

25. $p(x) = x^3 + 3x^2 - 25x + 21$

26. $h(x) = 10x^3 - 17x^2 - 7x + 2$

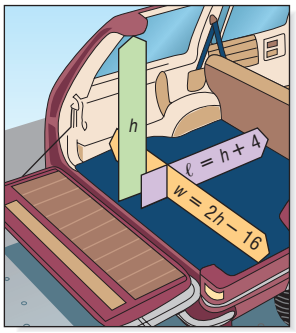
27. $g(x) = 48x^4 - 52x^3 + 13x - 3$

28. $p(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10$

29. $h(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12$

AUTOMOBILES **For Exercises 30 and 31, use the following information.**

The length of the cargo space in a sport-utility vehicle is 4 inches greater than the height of the space. The width is sixteen inches less than twice the height. The cargo space has a total volume of 55,296 cubic inches.



30. Use a rectangular prism to model the cargo space. Write a polynomial function that represents the volume of the cargo space.

31. Will a package 34 inches long, 44 inches wide, and 34 inches tall fit in the cargo space? Explain.

FOOD **For Exercises 32–34, use the following information.**

A restaurant orders spaghetti sauce in cylindrical metal cans. The volume of each can is about 160π cubic inches, and the height of the can is 6 inches more than the radius.

32. Write a polynomial equation that represents the volume of a can. Use the formula for the volume of a cylinder, $V = \pi r^2 h$.

33. What are the possible values of r? Which values are reasonable here?

34. Find the dimensions of the can.

AMUSEMENT PARKS **For Exercises 35–37, use the following information.**

An amusement park owner wants to add a new wilderness water ride that includes a mountain that is shaped roughly like a square pyramid. Before building the new attraction, engineers must build and test a scale model.

35. If the height of the scale model is 9 inches less than its length, write a polynomial function that describes the volume of the model in terms of its length. Use the formula for the volume of a pyramid, $V = \frac{1}{3}Bh$.

36. If the volume is 6300 cubic inches, write an equation for the situation.

37. What are the dimensions of the scale model?

EXTRA PRACTICE
See pages 905, 931.

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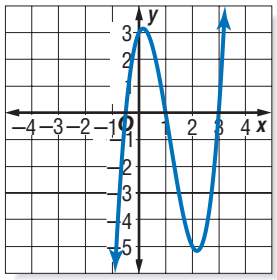
For Exercises 38 and 39, use the following information.

38. Find all of the zeros of $f(x) = x^3 - 2x^2 + 3$ and $g(x) = 2x^3 - 7x^2 + 2x + 3$.

39. Determine which function, f or g, is shown in the graph at the right.



H.O.T. Problems

40. FIND THE ERROR Lauren and Luis are listing the possible rational zeros of $f(x) = 4x^5 + 4x^4 - 3x^3 + 2x^2 - 5x + 6$. Who is correct? Explain your reasoning.

Lauren	Luis
$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}$	$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm\frac{3}{2}, \pm 3, \pm\frac{3}{4}, \pm 6$

41. OPEN ENDED Write a polynomial function that has possible rational zeros of $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$.

42. CHALLENGE If k and $2k$ are zeros of $f(x) = x^3 + 4x^2 + 9kx - 90$, find k and all three zeros of $f(x)$.

43. *Writing in Math* Use the information on page 369 to explain how the Rational Zero Theorem can be used to solve problems involving large numbers. Include the polynomial equation that represents the volume of the overhead baggage compartment and a list of all measures of the width of the compartment, assuming that the width is a whole number.

STANDARDIZED TEST PRACTICE

44. Which of the following is a zero of the function $f(x) = 12x^5 - 5x^3 + 2x - 9$?

A -6

B $\frac{3}{8}$

C $-\frac{2}{3}$

D 1

45. REVIEW A window is in the shape of an equilateral triangle. Each side of the triangle is 8 feet long. The window is divided in half by a support from one vertex to the midpoint of the side of the triangle opposite the vertex. Approximately how long is the support?

F 5.7 ft **H** 11.3 ft

G 6.9 ft **J** 13.9 ft

Spiral Review

Given a function and one of its zeros, find all of the zeros of the function. (Lesson 6-8)

46. $g(x) = x^3 + 4x^2 - 27x - 90; -3$

47. $h(x) = x^3 - 11x + 20; 2 + i$

48. $f(x) = x^3 + 5x^2 + 9x + 45; -5$

49. $g(x) = x^3 - 3x^2 - 41x + 203; -7$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. (Lesson 6-7)

50. $20x^3 - 29x^2 - 25x + 6; x - 2$

51. $3x^4 - 21x^3 + 38x^2 - 14x + 24; x - 3$

52. GEOMETRY The perimeter of a right triangle is 24 centimeters. Three times the length of the longer leg minus two times the length of the shorter leg exceeds the hypotenuse by 2 centimeters. What are the lengths of all three sides? (Lesson 3-5)